

Quantum spin liquid near Mott transition with fermionized π -vortices

Su-Peng Kou,^{1,*} Lan-Feng Liu,¹ Jing He,¹ and Ya-Jie Wu¹

¹*Department of Physics, Beijing Normal University, Beijing 100875, China*

In this paper, we study the non-magnetic insulator state near Mott transition of 2D π -flux Hubbard model on square lattice and find that such non-magnetic insulator state is quantum spin liquid state with nodal fermionic excitations - nodal spin liquid (NSL). When there exists small easy-plane anisotropic energy, the ground state becomes Z_2 topological spin liquid (TSL) with full gapped excitations. The $U(1) \times U(1)$ mutual-Chern-Simons (MCS) theory is obtained to describe the low energy physics of NSL and TSL.

PACS numbers: 74.20.Mn, 74.25.Ha, 75.10.-b

I. INTRODUCTION

People have been looking for quantum spin liquid states in frustrated spin models for more than 20 years^{1,2}. For example, various approaches show that quantum spin liquids may exist in two-dimensional (2D) $S = 1/2$ J_1 - J_2 model or the Heisenberg model on Kagomé lattice³⁻⁷. In these models, the quantum spin liquids are accessed (in principle) by appropriate frustrating interactions. However, the nature of the quantum disordered ground state is still much debated⁸⁻¹⁵. People have guessed that the quantum disordered state may be either algebra spin liquid state^{16,17}, or Z_2 spin liquid state¹⁸ or chiral spin liquid^{19,20}.

On the other hand, the experiments in the organic material $\kappa - (\text{BEDT} - \text{TTF})_2\text{Cu}_2(\text{CN})_3$ indicate the realization of the quantum spin liquid states²¹⁻²³. Then, $U(1)$ and $SU(2)$ slave-roton theories of the Hubbard model were formulated on the triangular or honeycomb lattices^{24,25}. Recently, the quantum spin liquid state near Mott transition of the Hubbard model on honeycomb lattice has been conformed by different approaches²⁶⁻³². In particular, in Ref.²⁶, quantum spin liquid state has been predicted by quantum Monte Carlo (QMC) simulation.

In Ref.³³, the possible non-magnetic state in the nodal antiferromagnetic (AF) insulator (an insulator with AF spin density wave ordering and massive Dirac fermionic excitations) is predicted to near the Mott transition of the π -flux Hubbard model (See Fig.1). Such type of quantum disordered states in bipartite lattices is also not driven by frustrations, as people have done in varied

spin models. Instead, they come from quantum fluctuations of a relatively small effective spin-moments near Mott transition. What's the nature of the possible non-magnetic state here? In this paper we will answer the question. We find that for the isotropic case, the non-magnetic state with $SU(2)$ spin rotation symmetry is a new type of spin liquid - *nodal spin liquid (NSL) with gapless fermionic excitations and roton-like excitations*; on the other hand, for the anisotropic case by adding small easy-plane anisotropic term, the non-magnetic state with weakly breaking $SU(2)$ spin rotation symmetry becomes a Z_2 topological spin liquid (TSL) with topological degenerate ground states.

The paper is organized as follows. In Sec.II, we study the spin fluctuations of the nodal AF insulator (NAI) in π -flux Hubbard model on square lattice based on a formulation by keeping spin rotation symmetry. In Sec.III, we study the properties of half skyrmions - topological solitons of the nodal AF insulator and show their induced quantum numbers and statistics. In Sec.IV, nodal spin liquid is proposed to be the ground state of the quantum non-magnetic insulator state in the nodal AF insulator. In this section, we use $U(1) \times U(1)$ mutual-Chern-Simons theory to learn the properties of NSL and TSL. Finally, the conclusions are given in Sec.V.

II. NON-MAGNETIC STATE IN NODAL AF INSULATOR OF THE HUBBARD MODEL ON π -FLUX LATTICE

In this paper we will focus on the π -flux Hubbard model on square lattice³⁵. The Hamiltonian of it is

$$\mathcal{H} = - \sum_{\langle i,j \rangle} (t_{ij} \hat{c}_i^\dagger \hat{c}_j + h.c.) + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}. \quad (1)$$

Here $\hat{c}_i = (\hat{c}_{i\uparrow}, \hat{c}_{i\downarrow})^T$ are defined as electronic annihilation operators. U is the on-site Coulomb repulsion. $\langle i,j \rangle$ denotes two sites on a nearest-neighbor link. $\hat{n}_{i\uparrow}$ and $\hat{n}_{i\downarrow}$ are the number operators of electrons at site i with up-spin and down-spin, respectively. There is a π -flux phase when a particle hops around a plaquette in a π -flux lattice. So the nearest-neighbor hopping $t_{i,j}$ in a π -flux lattice could be chosen as $t_{i,i+\hat{x}} = t, t_{i,i+\hat{y}} =$

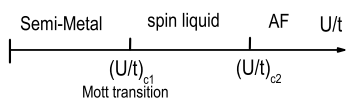


FIG. 1: The scheme of the spin liquid state near Mott transition of the π -flux Hubbard model

$te^{\pm i\frac{\pi}{2}}$ ³⁴. Although π -flux phase does not break translational symmetry, we may still divide the square lattice into two sublattices, A and B . After transforming the hopping term into momentum space, we obtain $\xi_{\mathbf{k}} = \pm \sqrt{4t^2 (\cos^2 k_x + \cos^2 k_y)}$. So there exist two nodal fermi-points at $\mathbf{k}_1 = (\frac{\pi}{2}, \frac{\pi}{2})$, $\mathbf{k}_2 = (\frac{\pi}{2}, -\frac{\pi}{2})$ and the spectrum of fermions becomes linear in the vicinity of the two nodal points. In the non-interacting limit, the π -flux Hubbard model is reduced into a free fermion model with nodal fermions at half filling.

Because the Hubbard model on bipartite lattices is unstable against antiferromagnetic instability, at half-filling, the ground state may be an insulator with AF SDW order with increasing interacting strength. Such AF SDW order is described by the following mean field order parameter $\langle (-1)^i \hat{c}_i^\dagger \sigma_z \hat{c}_i \rangle = \frac{M}{2}$. Here M is the staggered magnetization. Then in the mean field theory, the Hamiltonian of the 2D π -flux Hubbard model is obtained as

$$\mathcal{H} = - \sum_{\langle ij \rangle} (t_{i,j} \hat{c}_i^\dagger \hat{c}_j + h.c.) - \sum_i (-1)^i \Delta \hat{c}_i^\dagger \sigma_z \hat{c}_i \quad (2)$$

where $\Delta = \frac{UM}{2}$ leads to the energy gap of electrons and σ_z is the Pauli matrix. After diagonalization, the spectrum of the electrons is obtained as $E_{\mathbf{k}} = \pm \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$. By minimizing the free energy at zero temperature, the self-consistent equation of (2) is reduced into $\frac{1}{N} \sum_{\mathbf{k}} \frac{U}{2E_{\mathbf{k}}} =$

1 where N is the number of the particles.

From mean field approach, one may see that the MI transition of the π -flux Hubbard model occurs at a critical value about $U/t \simeq 3.11$. In the weakly coupling

limit ($U/t < 3.11$), the ground state is a semi-metal (SM) with nodal fermi-points. In the strong coupling region ($U/t > 3.11$), due to $M \neq 0$, the ground state becomes an insulator with relativistic massive fermionic excitations. However, the non-zero value of M only means the existence of effective spin moments. It does not necessarily imply that the ground state of NAI is a long range AF order in the mean field theory. Thus one needs to examine stability of magnetic order against quantum fluctuations of effective spin moments based on a formulation by keeping spin rotation symmetry, $\sigma_z \rightarrow \boldsymbol{\Omega} \cdot \boldsymbol{\sigma}$.

Within the haldane's mapping, the spins are parametrized as $\boldsymbol{\Omega}_i = (-1)^i \mathbf{n}_i \sqrt{1 - \mathbf{L}_i^2} + \mathbf{L}_i$ ³⁶⁻³⁹. Here \mathbf{n}_i is the AF order parameter and $|\mathbf{n}_i| = 1$, \mathbf{L}_i is the transverse canting field, which is chosen to $\mathbf{L}_i \cdot \mathbf{n}_i = 0$. By replacing the electronic operators \hat{c}_i^\dagger and \hat{c}_j by Grassmann variables c_i^* and c_j , we get the effective Lagrangian with spin rotation symmetry as

$$\mathcal{L}_{\text{eff}} = \sum_i c_i^* \partial_\tau c_i - \sum_{\langle ij \rangle} (t_{i,j} c_i^* c_j + h.c.) - \sum_i (-1)^i \Delta c_i^* \boldsymbol{\Omega}_i \cdot \boldsymbol{\sigma} c_i. \quad (3)$$

After integrating the massive fermions and transverse canting field, an effective O(3) nonlinear σ model appears that describes the long-wavelength spin fluctuations with spin rotation symmetry^{2,33,38-42}

$$\mathcal{L}_s = \frac{1}{2g} (\partial_\mu \mathbf{n})^2. \quad (4)$$

The coupling constant is obtained as $g = \sqrt{\frac{1}{\chi^\perp \rho_s}}$ where

$$\rho_s = \frac{1}{N} \sum_{\mathbf{k}} \frac{t^2 [\cos(2k_x) (\Delta^2 + 8t^2 + 4t^2 \cos(2k_y)) + \Delta^2 + 3t^2 + t^2 \cos(4k_x)]}{2(|\xi_{\mathbf{k}}|^2 + \Delta^2)^{\frac{3}{2}}} \quad (5)$$

and the transverse spin susceptibility is

$$\chi^\perp = [(\frac{1}{N} \sum_{\mathbf{k}} \frac{\Delta^2}{4(|\xi_{\mathbf{k}}|^2 + \Delta^2)^{\frac{3}{2}}})^{-1} - 2U]^{-1}. \quad (6)$$

See detailed calculations in appendix. In the paper we will set the spin velocity to be unit.

The coupling constant g of the π -flux Hubbard model has been obtained in Ref.³³. In Ref.³³, a quantum critical point at $g_c = \frac{4\pi}{\Lambda}$ was found that corresponds to $U/t \simeq 4.26$ of the π -flux Hubbard model. For the case of $g > g_c$, a quantum non-magnetic insulator (NMI) state with gapped spin excitations as $m_s = 4\pi(\frac{1}{g_c} - \frac{1}{g})$ appears that corresponds to the region of $3.11 < U/t < 4.26$; For the case of $g < g_c$, the ground state is a long range AF order that corresponds to the region of $U/t > 4.26$. Here the momentum cutoff is introduced as $\Lambda = \min(a^{-1},$

$2\Delta)$ ^{38,39} (a is lattice constant). The existence of a non-magnetic insulator state provides an alternative candidate for finding spin liquid state. An interesting issue is the nature of the non-magnetic insulator. Is it a valence-band crystal, or algebra spin liquid state, or a new type of quantum state? In the following parts we will answer the question.

III. HALF-SKYRMION AS FERMIONIC EXCITATION

To learn the nature of the non-magnetic insulator, one may study its excitations. In the non-magnetic insulator, we get the effective model of massive spin-1 excitations

$$\mathcal{L}_s = \frac{1}{2g} [(\partial_\mu \mathbf{n})^2 + m_s^2 \mathbf{n}^2]. \quad (7)$$

Or using the CP(1) representation, we have

$$\mathcal{L}_s = \frac{2}{g} [|(\partial_\mu - ia_\mu)\mathbf{z}|^2 + m_z^2 \mathbf{z}^2] \quad (8)$$

where \mathbf{z} is a bosonic spinon, $\mathbf{z} = (z_1, z_2)$, $\mathbf{n}_i = \bar{\mathbf{z}}_i \sigma \mathbf{z}_i$, $\bar{\mathbf{z}}\mathbf{z} = 1$, $a_\mu \equiv -\frac{i}{2}(\bar{\mathbf{z}}\partial_\mu \mathbf{z} - \partial_\mu \bar{\mathbf{z}}\mathbf{z})$. Here a_μ is introduced as an assistant gauge field. Because the bosonic spinons have mass gap, we may integrate them and get $\mathcal{L}_{\text{eff}} = \frac{1}{4e_a^2}(\partial_\mu a_\nu)^2$ with $e_a^2 \sim m_z = \frac{1}{2}m_s$. Due to the instanton effect, the gauge field a_μ obtains a mass gap and bosonic spinons that couple the gauge field a_μ are confined. Now the lowest energy excitations are gapped spin wave. However, the answer is not quite right. Applying the Oshikawa's commensurability condition and Hastings' theorem to the present case, the quantum state without spontaneously symmetry breaking can be either a topological order with degenerate ground state or a uniform ground state with triplet excitation gap must be accompanied with other gapless excitations^{47,48}. In this paper we indeed find that non-magnetic insulator is either topological spin liquid with degenerate ground state or a nodal spin liquid with gapless fermionic excitations.

A. half-skyrmion

In the following parts, we focus on the half-skyrmions (topological vortices with half topological charge),

$$\mathcal{Q} = \int d^2\mathbf{r} \frac{1}{4\pi} \epsilon_{0\nu\lambda} \mathbf{n} \cdot \partial^\nu \mathbf{n} \times \partial^\lambda \mathbf{n} = \pm \frac{1}{2}. \quad (9)$$

In the vector \mathbf{n} representation, the solutions of the half-skyrmion of the continuum limit are⁴⁹⁻⁵⁶

$$\mathbf{n}_{\text{hs}} = \left(\frac{\lambda(x-x_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, \pm \frac{\lambda(y-y_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, \right. \\ \left. \pm \frac{\lambda}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}} \right). \quad (10)$$

Here λ is the radius of the half-skyrmion at $\mathbf{r}_0 = (x_0, y_0)$. Inside the core $|\mathbf{r}-\mathbf{r}_0|^2 < \lambda$, the spin is polarized; outside it $|\mathbf{r}-\mathbf{r}_0|^2 > \lambda$, one gets a vortex-like spin configuration on XY plane. To stabilize topological vortices (half-skyrmions), a small easy-plane anisotropic term should be added to the original model phenomenally, $H' = \kappa \sum_i \mathbf{n}_i^2$ ($\kappa > 0$, $\frac{\kappa}{t} \ll 1$).

From the exact solutions, there exist two types of merons : one has a up-spin polarized core $\mathcal{Q} = -\frac{1}{2}$,

$$(\mathbf{n}_{\text{hs}})_1 = \left(\frac{\lambda(x-x_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, \frac{\lambda(y-y_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, \right. \\ \left. \frac{\lambda}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}} \right); \quad (11)$$

the other a down-spin polarized core $\mathcal{Q} = \frac{1}{2}$,

$$(\mathbf{n}_{\text{hs}})_2 = \left(\frac{\lambda(x-x_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, \frac{\lambda(y-y_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, \right. \\ \left. - \frac{\lambda}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}} \right). \quad (12)$$

Fig.2 and Fig.3 show the two types of merons. From them, one can see that the topological charge of a half-skyrmion is determined by *both the spin configuration and the polarized direction of AF order in the core*.

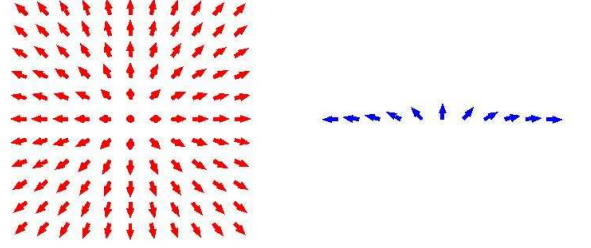


FIG. 2: The scheme of $(\mathbf{n}_{\text{hs}})_1$, the meron with topological charge $-1/2$. up-spins locate at the center.

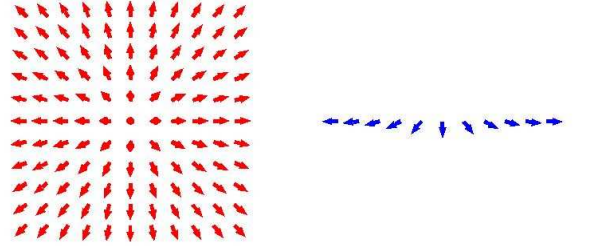


FIG. 3: The scheme of $(\mathbf{n}_{\text{hs}})_2$, the meron with topological charge $1/2$. Down-spins locate at the center.

Also, there are two types of anti-merons : one has a up-spin polarized core $\mathcal{Q} = \frac{1}{2}$,

$$(\mathbf{n}_{\text{hs}})_3 = \left(\frac{\lambda(x-x_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, - \frac{\lambda(y-y_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, \right. \\ \left. \frac{\lambda}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}} \right); \quad (13)$$

the other has a down-spin polarized core $\mathcal{Q} = -\frac{1}{2}$,

$$(\mathbf{n}_{\text{hs}})_4 = \left(\frac{\lambda(x-x_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, - \frac{\lambda(y-y_0)}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}}, \right. \\ \left. - \frac{\lambda}{\sqrt{|\mathbf{r}-\mathbf{r}_0|^2 + \lambda^2}} \right). \quad (14)$$

In AF ordered state, the mass of half-skyrmion m_{hs} is associated with the ordered staggered moment :

$$m_{hs} = 2\pi\tilde{\rho}_s + E_{\text{core}} \quad (g < g_c) \quad (15)$$

where $\tilde{\rho}_s = \rho_s(1 - \frac{g}{g_c})$ is the renormalized spin stiffness and E_{core} is the core energy. For the isotropic case $\kappa = 0$, the energy of the half-skyrmion does not depend on its radius λ and the core energy is zero, $E_{\text{core}} = 0$. For the anisotropy case $\kappa > 0$, the scale invariance is broken at $H' = \kappa \sum_i \mathbf{n}_i^2 \sim H_{\text{kinetic}}$ where H_{kinetic} is the kinetic energy, $H_{\text{kinetic}} = \frac{\Lambda}{2g}(\nabla \mathbf{n})^2$. A new scale $l^2 = \frac{a^2 \Lambda}{2g\kappa}$ appears (a is lattice constant). Now, the core energy is estimated as $E_{\text{core}} = \text{const} \cdot (\frac{\lambda}{l})^2$ which indicates the instability of the static soliton against collapse, $\lambda \rightarrow 0$. The half-skyrmion without a core $\lambda \rightarrow 0$ turns into a spin-vortex as

$$\mathbf{n}_{\text{hs}} = \left(\frac{x - x_0}{\sqrt{|\mathbf{r} - \mathbf{r}_0|^2 + \Lambda^{-1}}}, \pm \frac{y - y_0}{\sqrt{|\mathbf{r} - \mathbf{r}_0|^2 + \Lambda^{-1}}}, 0 \right). \quad (16)$$

See Fig.4, all spins of the spin-vortex are suppressed onto the XY plane. On the other hand, for the easy-axis anisotropic case, $\kappa < 0$, the energy of spin-vortex will diverge, $m_{hs} \sim N\kappa \rightarrow \infty$.

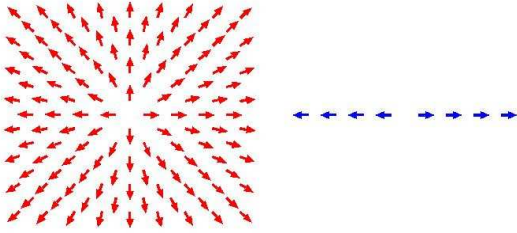


FIG. 4: The scheme of the spin-vortex. All spins are suppressed onto the XY plane.

B. Fermionic zero modes

In this part we calculate the fermionic zero modes around the spin-vortex. By replacing the operator \hat{c}_i and site number i by Grassmann number $\psi(x)$ and continuum coordinates x, y , we calculate the fermion bound state on the spin-vortex by the continuum formula of the following effective Hamiltonian

$$\mathcal{H}_{\text{eff}} = - \sum_{\langle ij \rangle} (t_{i,j} \hat{c}_i^\dagger \hat{c}_j + h.c.) + \sum_i (-1)^i \Delta \hat{c}_i^\dagger \mathbf{n}_{\text{hs}} \cdot \sigma \hat{c}_i \quad (17)$$

In the continuum limit, the two-flavor Dirac-like effective Lagrangian^{2,58-60} describes the low energy fermionic excitations at two nodes $\mathbf{k}_1 = (\frac{\pi}{2}, \frac{\pi}{2})$, $\mathbf{k}_2 = (\frac{\pi}{2}, -\frac{\pi}{2})$,

$$\mathcal{L}_{\text{eff}} = i\bar{\psi}_1 \gamma_\mu \partial_\mu \psi_1 + i\bar{\psi}_2 \gamma_\mu \partial_\mu \psi_2 + m_e (\bar{\psi}_1 \mathbf{n}_{\text{hs}} \cdot \sigma \psi_1 - \bar{\psi}_2 \mathbf{n}_{\text{hs}} \cdot \sigma \psi_2) \quad (18)$$

where $\bar{\psi}_1 = \psi_1^\dagger \gamma_0 = (\bar{\psi}_{\uparrow 1A}, \bar{\psi}_{\uparrow 1B}, \bar{\psi}_{\downarrow 1A}, \bar{\psi}_{\downarrow 1B})$ and $\bar{\psi}_2 = \psi_2^\dagger \gamma_0 = (\bar{\psi}_{\uparrow 2B}, \bar{\psi}_{\uparrow 2A}, \bar{\psi}_{\downarrow 2B}, \bar{\psi}_{\downarrow 2A})$. γ_μ is defined as $\gamma_0 = \sigma_0 \otimes \tau_z$, $\gamma_1 = \sigma_0 \otimes \tau_y$, $\gamma_2 = \sigma_0 \otimes \tau_x$, $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. τ^x, τ^y, τ^z are Pauli matrices. m_e denoted the mass gap of the electrons. We have set the Fermi velocity to be unit $v_F = 1$. The solution of zero modes has given by

$$\psi_1^0(\mathbf{r}) = \begin{pmatrix} 0 \\ \exp(-\frac{|\mathbf{r}-\mathbf{r}_0|}{m_e}) \\ \exp(-\frac{|\mathbf{r}-\mathbf{r}_0|}{m_e}) \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_2^0(\mathbf{r}) = \begin{pmatrix} 0 \\ -\exp(-\frac{|\mathbf{r}-\mathbf{r}_0|}{m_e}) \\ \exp(-\frac{|\mathbf{r}-\mathbf{r}_0|}{m_e}) \\ 0 \end{pmatrix}$$

in Ref.^{56,61}.

On the other hand, the double zero modes of a spin-vortex on a 21×21 lattice was shown in Fig.5. The exact charge is localized around the defect center within a length-scale $\sim \Delta^{-1}$. For the two spin-vortices, the fermion zero modes are slightly split due to tunneling between them. From numerical results, we find that there exist two zero modes on each spin-vortex. The fermion zero modes around a π spin-vortex match the results in^{56,61}.

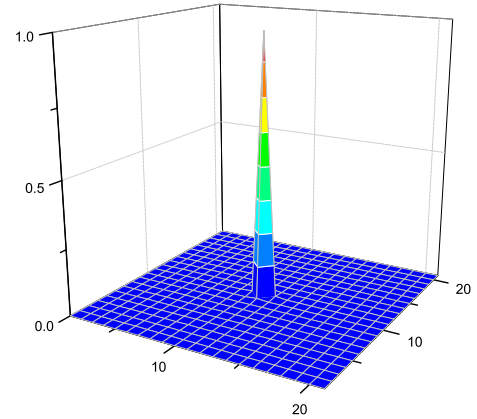


FIG. 5: The fermionic zero mode of spin-vortex on a 21×21 lattice.

C. Induced quantum numbers and statistics

Next, we calculate the induced quantum number on the spin-vortices and show the relationship between the induced quantum numbers and the topological charge of them⁶².

For the solutions of zero modes, there are four zero-energy soliton states around a spin-vortex which are denoted by

$$\begin{aligned} & |a_1\rangle \otimes |b_1\rangle, \quad |a_0\rangle \otimes |b_0\rangle, \\ & |a_0\rangle \otimes |b_1\rangle, \quad |a_1\rangle \otimes |b_0\rangle. \end{aligned} \quad (19)$$

Here $|a_0\rangle$ and $|b_0\rangle$ are empty states of the zero modes $\psi_1^0(\mathbf{r})$ and $\psi_2^0(\mathbf{r})$; $|a_1\rangle$ and $|b_1\rangle$ are occupied states of them. Without doping, the soliton states of a spin-vortex $|\text{sol}\rangle$ are denoted by $|a_0\rangle \otimes |b_1\rangle$ and $|a_1\rangle \otimes |b_0\rangle$.

In Ref.⁵⁶, the induced quantum numbers on the solitons states including total induced fermion number $\hat{N}_F = \sum_{\alpha,i} \hat{c}_i^\dagger \sigma_z \hat{c}_i$ and the induced staggered spin number $\hat{S}_{(\pi,\pi)}^z = \frac{1}{2} \sum_{i \in A} \hat{c}_i^\dagger \sigma_z \hat{c}_i - \frac{1}{2} \sum_{i \in B} \hat{c}_i^\dagger \sigma_z \hat{c}_i$ has been calculated. The total induced fermion number on the solitons is zero from the cancelation effect between two nodals $\hat{N}_F |\text{sol}\rangle = 0$. However, there exists an induced staggered spin moment on the soliton states⁵⁶, $\hat{S}_{(\pi,\pi)}^z |\text{sol}\rangle = \pm \frac{1}{2} |\text{sol}\rangle$.

In particular, we will show the relationship between the induced staggered spin moment $S_{(\pi,\pi)}^z$ and the topological charge of the spin-vortex \mathcal{Q} . Remember we have study the zero modes and the induced quantum number on a spin-vortex. Now the spin polarization inside the core of the spin-vortex will be determined by the induced spin moment : for the case of $S_{(\pi,\pi)}^z = \frac{1}{2}$, one has a up-spin polarized core, $\mathcal{Q} = -\frac{1}{2}$; for the case of $S_{(\pi,\pi)}^z = -\frac{1}{2}$, one has a down-spin polarized core, $\mathcal{Q} = \frac{1}{2}$. *As a result, the spin-vortex with induced staggered spin moment becomes a half-skyrmion with a narrow core and the topological charge of such half-skyrmion depends on the induced spin moment* (See Fig.6).



FIG. 6: The spin-vortex with trapped spin moment turns into a half skyrmion. The trapped spin moment is denoted by the red arrow.

On the other hand, since each bosonic spinon \mathbf{z} carries $\frac{1}{2}$ staggered spin moment, an induced staggered spin moment corresponds to a trapped bosonic spinon \mathbf{z} . So there is a mutual semion statistics between "bosonic spinon" \mathbf{z} and the half skyrmion with narrow core (meron or antimeron). Due to the mutual semion statistics, by binding the trapped bosonic spinon \mathbf{z} , a mobile half-skyrmion becomes a composite fermionic particle.

In order to giving a clear comparison we show these quantum numbers of a spin vortex (half-skyrmion with narrow core) in Table.1.

	$ a_1\rangle \otimes b_0\rangle$	$ a_0\rangle \otimes b_1\rangle$
Total fermion number	0	0
staggered spin number	$-\frac{1}{2}$	$\frac{1}{2}$
Topological charge	$\frac{1}{2}$	$-\frac{1}{2}$

In the long range AF ordered state ($g < g_c$), the spin-vortex will be confined and the total energy of it diverges. On the other hand, in the region of $g > g_c$, the spin-vortex has finite energy that is $E_{\text{core}} = \kappa$. Thus in the quantum non-magnetic insulator state ($g > g_c$), the mass of half-skyrmion vanishes in the isotropic limit⁵⁷, $m_{hs} = 0$ and survives when there is a small easy-plane anisotropic term $m_{hs} = E_{\text{core}} = \kappa$. For the easy-axis anisotropic case, $\kappa < 0$, the energy of spin-vortex always diverge, $m_{hs} \rightarrow \infty$.

IV. QUANTUM SPIN LIQUID STATE

Whether the non-magnetic insulator is a VBC state? Because the VBC state is characterized by the condensation of the half-skyrmions, if half-skyrmions are bosons, they will condense. Then one gets a VBC state (or quantum dimer state)^{63,64}. However, in the non-magnetic insulator half-skyrmion obeys fermionic statistics by trapping a bosonic spinon \mathbf{z} onto its core. The massless fermionic vortices lead to a new story of the quantum spin liquid states. Consequently, a new type of quantum spin liquid state - nodal spin liquid (NSL) is explored. In addition, after adding a small easy-plane anisotropic term, the fermionic vortices get energy gap, then the ground state becomes a topologically ordered spin liquid.

A. Effective model of half-skyrmions

Because the spin-vortices may have zero energy, to learn the low energy physics of non-magnetic insulator, we need to know quantum dynamics of them. In the following parts we will consider the spin-vortex as a quantum object and obtain its effective model.

As shown in Fig.6, the center of a spin-vortex is the plaquette rather than the original sites. So we may define dual lattice by I . The spin-vortex will hop from one dual lattice to another. Thus the spin-vortices show similar behavior of vortices in XY model : it can move on dual lattice with the same lattice constant and feel an effective π -flux phase^{51,52} (See detail in Appendix C).

We may use the operator $f_{I\sigma}$ to describe such neutral fermionic particle with half spin at dual lattice I . Here σ is the spin index. The relation between the zero energy states and the fermionic states is given as

$$|a_1\rangle \otimes |b_0\rangle = f_{I\downarrow}^\dagger |0\rangle_f \quad (20)$$

and

$$|a_0\rangle \otimes |b_1\rangle = f_{I\uparrow}^\dagger |0\rangle_f \quad (21)$$

(The state $|0\rangle_f$ is defined through $f_{I\uparrow}|0\rangle_f = f_{I\downarrow}|0\rangle_f = 0$). We call such neutral object at dual lattice I (fermion with $\pm \frac{1}{2}$ spin degree freedom) a "fermionic spinon". Then the leading order of the hopping term of the half-skyrmions

is

$$\mathcal{H}_{hs} = - \sum_{\langle I, J \rangle} \left(\tilde{t}_{I, J} f_I^\dagger f_J + h.c. \right) + \sum_I m_{hs} (-1)^I f_I^\dagger f_I \quad (22)$$

where $f_I = (f_{I\uparrow}, f_{I\downarrow})^T$ are defined as fermionic spinon's annihilation operators. I and J denote two nearest-neighbor dual sites. Here $\tilde{t}_{I, J}$ is the effective hopping of the fermionic spinons. For the π -flux phase for the half-skyrmions, one also needs to divide the dual-lattice into two sublattices, A and B . After transforming the hopping term into momentum space, we obtain a d-wave like dispersion of the fermionic spinons $\epsilon_{\mathbf{k}} = \pm \sqrt{4\tilde{t}^2 (\cos^2 p_x + \cos^2 p_y)}$. So there also exist two nodal fermi-points at $\mathbf{p}_1 = (\frac{\pi}{2}, \frac{\pi}{2})$, $\mathbf{p}_2 = (\frac{\pi}{2}, -\frac{\pi}{2})$ and the spectrum of fermions spinons becomes linear in the vicinity of the two nodal points.

In the continuum limit, we get a Dirac-like effective Lagrangian that describes the low energy fermionic spinons

$$\mathcal{L}_{hs} = \sum_a i \bar{\Psi}_a \gamma_\mu \partial_\mu \Psi_a + i m_{hs} \sum_a \bar{\Psi}_a \Psi_a \quad (23)$$

where $\bar{\Psi}_1 = \Psi_1^\dagger \gamma_0 = (\bar{f}_{\uparrow 1A}, \bar{f}_{\uparrow 1B}, \bar{f}_{\downarrow 1A}, \bar{f}_{\downarrow 1B})$ and $\bar{\Psi}_2 = \Psi_2^\dagger \gamma_0 = (\bar{f}_{\uparrow 2B}, \bar{f}_{\uparrow 2A}, \bar{f}_{\downarrow 2B}, \bar{f}_{\downarrow 2A})$. γ_μ is defined as $\gamma_0 = \sigma_0 \otimes \tau_z$, $\gamma_1 = \sigma_0 \otimes \tau_y$, $\gamma_2 = \sigma_0 \otimes \tau_x$, $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. τ^x, τ^y, τ^z are Pauli matrices. For simplicity, we set the Fermi velocity to be unit. Thus in the isotropic limit $\kappa \rightarrow 0$, we may ignore the mass gap of Ψ_a and consider the fermionic spinons as gapless particles.

B. Mutual Chern-Simons theory

From above results, there exist two types of fields, the bosonic spinon \mathbf{z} and the fermionic spinon f_σ^\dagger . Now the low energy effective model becomes

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{hs} + \mathcal{L}_s = \frac{1}{2g} (\partial_\mu \mathbf{n})^2 + i \sum_{a\sigma} \bar{\Psi}_{a\sigma} (\gamma_\mu \partial_\mu + m_{hs}) \Psi_{a\sigma}. \quad (24)$$

However, there exists non-trivial topological relationship between \mathbf{z} and $\Psi_{a\sigma}$ - the fields $\Psi_{a\sigma}$ that carry $\pm \frac{1}{2}$ winding number of AF vector \mathbf{n} ,

$$\frac{1}{8\pi} \epsilon_{\mu\nu\lambda} \mathbf{n} \cdot \partial_\nu \mathbf{n} \times \partial_\lambda \mathbf{n} = -i \sum_a \bar{\Psi}_a \frac{\sigma^z}{2} \gamma_\mu \Psi_a. \quad (25)$$

Here the operator $\frac{\sigma^z}{2}$ means that the topological charge of the fermionic spinons is spin-dependence: For the meron with $\frac{1}{2}$ ($-\frac{1}{2}$) spin number, the topological charge $Q = -\frac{1}{2}$ ($Q = \frac{1}{2}$) with up-spin (down-spin) polarized in the core.

To ensure such constraint in Eq.(25), we add a new term in the effective Lagrangian,

$$\mathcal{L}_{\text{constraint}} = i A_\mu \left(\frac{1}{8\pi} \epsilon_{\mu\nu\lambda} \mathbf{n} \cdot \partial_\nu \mathbf{n} \times \partial_\lambda \mathbf{n} + i \sum_a \bar{\Psi}_a \frac{\sigma^z}{2} \gamma_\mu \Psi_a \right). \quad (26)$$

Finally the low energy effective theory is obtained as

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \mathcal{L}_{hs} + \mathcal{L}_s + \mathcal{L}_{\text{constraint}} \\ &= i \sum_a \bar{\Psi}_a \left(\gamma_\mu \partial_\mu + i \frac{\sigma^z}{2} \gamma_\mu A_\mu + m_{hs} \right) \Psi_a + \frac{1}{2g} (\partial_\mu \mathbf{n})^2 + \mathcal{L}_{MCS} \\ &= i \sum_a \bar{\Psi}_a \left(\gamma_\mu \partial_\mu + i \frac{\sigma^z}{2} \gamma_\mu A_\mu + m_{hs} \right) \Psi_a + \frac{2}{g} |(\partial_\mu - i a_\mu) \mathbf{z}|^2 + \mathcal{L}_{MCS}. \end{aligned} \quad (27)$$

In particular, there exists a mutual-Chern-Simons term,

$$\mathcal{L}_{MCS} = \frac{i}{\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda = \frac{i}{\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda. \quad (28)$$

To obtain above low energy effective theory, we have use the equation $\partial_\nu a_\lambda - \partial_\lambda a_\nu = \frac{1}{2} \mathbf{n} \cdot \partial_\nu \mathbf{n} \times \partial_\lambda \mathbf{n}$.

Then the effective model becomes an $U(1) \times U(1)$ mutual-Chern-Simons (MCS) gauge theory. Fermions $\Psi_{a\sigma}$ couple to an $U(1)$ gauge field A_μ ; boson spinon \mathbf{z} couples to an $U(1)$ gauge field a_μ . The results say that $\Psi_{a\sigma}$ act as half topological vortices for a_μ and \mathbf{z} act as half topological vortices of A_μ . This effective Lagrangian proposed in here and earlier papers^{65,66} retains

the full symmetries of translation, parity, time-reversal, and global spin rotation, in contrast to the conventional Chern-Simons theories where the second and third symmetries are usually broken. In the following parts of the paper, we use $U(1) \times U(1)$ MCS theory to learn the quantum non-magnetic state near Mott transition.

C. AF order

If $g < g_c$, the ground state has a long range Néel order, $\langle \mathbf{z} \rangle = \mathbf{z}_0 \neq 0$ or $\langle \mathbf{n} \rangle \neq 0$. Hence the effective model turns

into

$$\mathcal{L}_{\text{eff}} \simeq \frac{2z_0^2}{g_{\text{eff}}} |a_\mu|^2 + \frac{i}{\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda + i \sum_a \bar{\Psi}_a \left(\gamma_\mu \partial_\mu + i \frac{\sigma^z}{2} \gamma_\mu A_\mu + m_{hs} \right) \Psi_a \quad (29)$$

where g_{eff} is renormalized coupling constant as $g_{\text{eff}} = \frac{g \cdot g_c}{(g_c - g)}$. Here the mass term of the gauge field a_μ is caused by \mathbf{z} condensation. After integrating a_μ we get a renormalized kinetic term for gauge field A_μ ,

$$\mathcal{L}_{\text{eff}} = \frac{1}{4e_A^2} (\partial_\mu A_\nu)^2 + \mathcal{L}_{hs} \quad (30)$$

with $\frac{1}{e_A^2} = \frac{g_{\text{eff}}}{z_0^2 \pi^2}$. Due to the instanton effect, the gauge field A_μ also obtains a mass gap and a linear confinement appears for fermions⁶⁷. The low energy physics is dominated only by spin waves and the effective Lagrangian becomes $\mathcal{L}_{\text{eff}} = \frac{1}{2g_{\text{eff}}} (\partial_\mu \mathbf{n})^2$.

At small T , the AF order is believed in the so-called renormalized classical (RC) region. In the RC region, the effective Lagrangian loses the Lorentz invariance and becomes $\mathcal{L}_{\text{eff}} \rightarrow \frac{1}{2} \bar{\rho}_s (\nabla \mathbf{n})^2$. Here $\bar{\rho}_s$ is the renormalized spin stiffness as $\bar{\rho}_s = \rho_s (1 - \frac{g_c}{g})$. Then at low temperatures, fermionic vortices are paired by the logarithmic-attractive interaction $V(\mathbf{r}) = 2\pi \bar{\rho}_s \ln \frac{|\mathbf{r}|}{a_0}$. With the increase of temperature, neutral vortex-antivortex pairs like those in the XY model are thermally excited, leading to a conventional contribution to the screening effect. By the conventional Kosterlitz-Thouless (KT) theory, there exists a “deconfining” temperature, $T_{\text{de}} \simeq \frac{\pi \bar{\rho}_s}{2}$. Above T_{de} , free excited fermionic vortices exist.

D. Nodal spin liquid - isotropic case $\kappa = 0$

If $g > g_c$, the ground state has no AF long range order. Now \mathbf{z} has a mass gap m_z , or one has the massive spin-1 quanta,

$$\mathcal{L}_s = \frac{1}{2g} \left[|(\partial_\mu - ia_\mu) \mathbf{z}|^2 + m_z^2 \mathbf{z}^2 \right]. \quad (31)$$

For the isotropic case $\kappa = 0$, the fermionic vortex has no energy gap. Thus the effective model becomes

$$\mathcal{L}_{\text{eff}} = i \sum_a \bar{\Psi}_a \left(\gamma_\mu \partial_\mu + i \frac{\sigma^z}{2} \gamma_\mu A_\mu \right) \Psi_a + \frac{1}{2g} \left[|(\partial_\mu - ia_\mu) \mathbf{z}|^2 + m_z^2 \mathbf{z}^2 \right] + \frac{i}{\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda. \quad (32)$$

After integrating the massless fermions and the massive bosonic spinons, the effective model turns into

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4e_a^2} (\partial_\nu a_\mu)^2 - \frac{1}{4e_A^2} (\partial_\nu A_\mu)^2 + \frac{i}{\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda \quad (33)$$

where $e_a^2 \simeq 3\pi m_z$, $\frac{1}{4e_A^2} = \frac{1}{4\sqrt{p^2}}$ and $p^2 = p_\mu^2$ is the momentum. Are bosonic spinons and fermionic vortices real quasi-particle? To answer the question, we need to calculate the energy gap of the gauge fields furthermore.

Firstly, after integrating a_μ we obtain a mass term for gauge field A_μ ,

$$\mathcal{L}_{\text{eff}} = i \sum_a \bar{\Psi}_a \left(\gamma_\mu \partial_\mu + i \frac{\sigma^z}{2} \gamma_\mu A_\mu \right) \Psi_a - \frac{1}{4e_A^2} (\partial_\nu A_\mu)^2 + \frac{e_a^2}{\pi^2} A_\mu^2. \quad (34)$$

Due to exchanging the gauge field A_μ , a short range interaction is induced between fermions. It is obvious that the short range interaction is irrelevant. As a result, the massless Dirac particles Ψ_a that couple to A_μ become real low energy degrees of freedom.

Secondly, after integrating A_μ we get the effective model of a_μ as the

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4e_a^2} (\partial_\nu a_\mu)^2 + \frac{e_A^2}{\pi^2} a_\mu^2. \quad (35)$$

This Lagrangian has the schematic form $-\frac{1}{2e_a^2} p^2 + \frac{1}{\pi^2} p$. That means the momentum of the gauge field a_μ is not zero, $p = \frac{2e_a^2}{\pi^2} \neq 0$. Then the gauge field a_μ has a finite energy gap ($\mathcal{L}_{\text{eff}} \sim \frac{2e_a^2}{\pi^4} a_\mu^2$) and shows roton-like behavior. As a result, the induced interaction by exchanging the gauge field a_μ is irrelevant and the bosonic spinons are real quasi-particles.

Finally, the low energy effective theory of nodal spin liquid becomes

$$\mathcal{L}_{\text{eff}} = i \sum_a \bar{\Psi}_a \gamma_\mu \partial_\mu \Psi_a + \frac{1}{2g} \left[|(\partial_\mu \mathbf{z})^2 + m_z^2 \mathbf{z}^2 \right] - \frac{1}{4e_a^2} (\partial_\nu a_\mu)^2 + \frac{\sqrt{p^2}}{\pi^2} a_\mu^2. \quad (36)$$

There are three types of quasi-particles: *two flavor gapless fermionic spinons* $\Psi_{a\sigma}$, *gapped bosonic spinons* \mathbf{z} and *the roton-like gauge fields* a_μ . Therefore, from the effective theory we conclude that NSL state is stable and can be considered as a new type of quantum spin liquid.

E. Topological spin liquid - anisotropic case $\kappa > 0$

For the anisotropic case $\kappa \neq 0$, the fermionic vortex has finite energy gap. Thus the effective model becomes

$$\mathcal{L}_{\text{eff}} = i \sum_a \bar{\Psi}_a \left(\gamma_\mu \partial_\mu + i \frac{\sigma^z}{2} \gamma_\mu A_\mu + m_{hs} \right) \Psi_a + \frac{1}{2g} \left[|(\partial_\mu - ia_\mu) \mathbf{z}|^2 + m_z^2 \mathbf{z}^2 \right] + \frac{i}{\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda. \quad (37)$$

After integrating the bosonic spinons \mathbf{z} and fermionic spinons Ψ_a , there appear the kinetic terms for gauge field a_μ and A_μ ,

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4e_a^2} (\partial_\nu A_\mu)^2 - \frac{1}{4e_A^2} (\partial_\mu a_\nu)^2 + \frac{i}{\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \quad (38)$$

where $e_a^2 \simeq 3\pi m_z$ and $e_A^2 \simeq \frac{3}{4}\pi m_{hs}$.

In particular, one may find that due to the mutual CS term, the mass term of one gauge field can be obtained by integrating the other : by integrating a_μ , the effective model of A_μ becomes

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4e_A^2}(\partial_\nu A_\mu)^2 + \frac{e_A^2}{\pi^2}A_\mu^2.$$

On the other hand, by integrating A_μ we get the effective model of a_μ as

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4e_a^2}(\partial_\nu a_\mu)^2 + \frac{e_a^2}{\pi^2}a_\mu^2.$$

Thus the gauge fields have mass gap

$$m_a = m_A = \frac{e_A e_a}{2\pi}.$$

Thus we find that the mutual $U(1) \times U(1)$ CS theory describes a topological ordered spin liquid without gapless excitations. Due to exchanging the gauge fields A_μ and a_μ , a short range interaction is induced between fermionic spinons and bosonic spinons. It is obvious that the short range interaction is irrelevant. As a result, the spinons that couple to A_μ or a_μ become real low energy degrees of freedom.

To show the topological properties of the topological spin liquid, we calculate ground state degeneracy on a torus. For periodic boundary condition, we can expand the gauge fields as

$$(A_x, A_y) = \left(\frac{1}{L_x} \Theta_x + \sum_{\mathbf{k}} A_{\mathbf{k}}^x e^{i\tilde{x} \cdot \mathbf{k}}, \frac{1}{L_y} \Theta_y + \sum_{\mathbf{k}} A_{\mathbf{k}}^y e^{i\tilde{x} \cdot \mathbf{k}} \right), \quad (39)$$

$$(a_x, a_y) = \left(\frac{1}{L_x} \theta_x + \sum_{\mathbf{k}} a_{\mathbf{k}}^x e^{i\tilde{x} \cdot \mathbf{k}}, \frac{1}{L_y} \theta_y + \sum_{\mathbf{k}} a_{\mathbf{k}}^y e^{i\tilde{x} \cdot \mathbf{k}} \right) \quad (40)$$

where $\mathbf{k} = (k_x, k_y) = (\frac{2\pi}{L_x} n_x, \frac{2\pi}{L_y} n_y)$ where $n_{x,y}$ are integers. $(A_{\mathbf{k}}^x, A_{\mathbf{k}}^y)$ and $(a_{\mathbf{k}}^x, a_{\mathbf{k}}^y)$ are the gauge fields with non-zero momentum and (Θ_x, Θ_y) and (θ_x, θ_y) are the zero modes with zero momentum for the gauge fields A_i and a_i . Because the existence of the mass gap, the degree freedoms for gauge fields with non-zero momentum $(A_{\mathbf{k}}^x, A_{\mathbf{k}}^y)$ and $(a_{\mathbf{k}}^x, a_{\mathbf{k}}^y)$ have nothing to do with the low energy physics. The low energy physics is determined by (Θ_x, Θ_y) and (θ_x, θ_y) .

For the temporal gauge $A_0 = 0$, after the mode expansion, we write down the following effective Hamiltonian to describe the low energy physics of the mutual $U(1) \times U(1)$ CS theory

$$H_{\text{eff}} = \frac{(P_{\Theta_x} - \frac{\theta_y}{\pi})^2}{2M_x} + \frac{p_{\theta_y}^2}{2m_y} + \frac{(p_{\theta_x} - \frac{\Theta_y}{\pi})^2}{2m_x} + \frac{P_{\Theta_y}^2}{2M_y}$$

where the conjugate momentum for (Θ_x, Θ_y) and (θ_x, θ_y) are defined as

$$P_{\Theta_x} = M_x \dot{\Theta}_x + \frac{\theta_y}{2\pi}, \quad P_{\Theta_y} = M_y \dot{\Theta}_y - \frac{\theta_x}{2\pi}$$

and

$$p_{\theta_x} = m_x \dot{\theta}_x + \frac{\Theta_y}{2\pi}, \quad p_{\theta_y} = m_y \dot{\theta}_y - \frac{\Theta_x}{2\pi}.$$

The masses are given as $M_x = \frac{1}{e_A^2} \frac{L_y}{L_x}$, $M_y = \frac{1}{e_A^2} \frac{L_x}{L_y}$ and $m_x = \frac{1}{e_a^2} \frac{L_y}{L_x}$, $m_y = \frac{1}{e_a^2} \frac{L_x}{L_y}$. Thus we map the original mutual $U(1) \times U(1)$ CS theory to a quantum mechanics model of two particles in two dimensions. The effective Hamiltonian of (Θ_x, Θ_y) and (θ_x, θ_y) corresponds to two particles on a torus through two-unit flux : (Θ_x, θ_y) are the coordinates of the first particle, and (Θ_y, θ_x) are the coordinates of the second particle. The degeneracy for (Θ_x, θ_y) degrees of freedom and the degeneracy for (Θ_y, θ_x) degrees of freedom are given as $D_{(\Theta_x, \theta_y)} = 2$ and $D_{(\Theta_y, \theta_x)} = 2$. As a result, for the mutual $U(1) \times U(1)$ CS theory, the ground states have four-fold degeneracy. That means the ground state of the anisotropic case $\kappa > 0$ is a Z_2 topological spin liquid.

For the easy-axis anisotropic case, $\kappa < 0$, the situation changes. For this case, the effective nonlinear σ model is not available for the temperature below the energy scale of the anisotropic κ . Thus the ground state are always long range (Ising) AF order in the insulator phase $M \neq 0$.

F. Experimental predictions

Firstly, we discuss the experimental predictions in the NSL.

In NSL, the spin-correlation decays exponentially

$$\langle S^+(x, y) S^-(0) \rangle = e^{i\mathbf{Q} \cdot \mathbf{R}_i} \langle n^+(x, y) n^-(0) \rangle \sim e^{i\mathbf{Q} \cdot \mathbf{R}_i} e^{-r/\xi} \quad (41)$$

with $\mathbf{Q} = (\pi, \pi)$, $r = \sqrt{x^2 + y^2}$ and $n^\pm = n^x \pm i n^y$. Here ξ is spin correlated length, $\xi = \frac{8\pi}{\Lambda_{\text{geff}}}$. In contrast, in algebraic spin liquid or algebraic vortex liquid, the spin-correlation shows critical behavior.

Secondly, we calculate the special heat. Because the fermionic spinons have no energy gap, the special heat is dominated by them. So at low temperature, the special heat is

$$C_V = \frac{12\zeta}{\pi} k_B^2 T^2 \quad (42)$$

where $\zeta = \int_0^\infty \frac{x^2 dx}{e^x + 1} = \frac{3}{4} \Gamma(3) \zeta(3) \simeq 1.803$.

Thirdly, we calculate the spin susceptibility. The definition of the spin susceptibility is

$$F = F(B=0) - \frac{1}{2} \chi B^2 \dots \quad (43)$$

where F is free energy and B is the external magnetic field. There are two contributions to the total spin susceptibility χ , one from bosonic spinons, the other is from the fermionic spinons as

$$\chi = \chi_b + \chi_f. \quad (44)$$

Here χ_b is given by^{46,68}

$$\chi_b = \frac{2}{3}\chi_{\perp}\mu_B^2 M^2 + \frac{2(2\mu_B M)^2}{\pi\beta} \left[\frac{m_s\beta}{1 - e^{-m_s\beta}} - \ln(e^{m_s\beta} - 1) \right] \quad (45)$$

with $\beta \equiv 1/k_B T$. The contribution from the fermionic spinons is almost linear temperature dependence as

$$\chi_f = \frac{4 \ln 2}{\pi\beta} \quad (46)$$

in unit of $(g\mu_B)^2$.

On the other hand, for TSL, the spin-correlation also decays exponentially $\langle S^+(x, y) S^-(0) \rangle \sim e^{i\mathbf{Q}\cdot\mathbf{R}_i} e^{-r/\xi}$ with $\xi = \frac{8\pi}{\Lambda g_{\text{eff}}}$. However, due to mass gap, $m_{hs} \neq 0$, at low temperature, the specific heat and the spin susceptibility from the fermionic spinons are all proportion to $e^{-\beta m_{hs}}$ and disappear at zero temperature.

V. CONCLUSION

Let us draw a conclusion. In this paper, we study the non-magnetic insulator state near Mott transition of 2D π -flux Hubbard model on square lattice and find that for the isotropic case such non-magnetic insulator state is quantum spin liquid state with nodal fermionic excitations - NSL; for the anisotropy case it is TSL with full gapped excitations. The low energy physics is basically determined by its $U(1) \times U(1)$ mutual Chern-Simons gauge theory. There exist both fermionic spinons and bosonic spinons. And it is just the mutual semion statistics between fermionic spinons and bosonic spinons that guarantee the stability of quantum spin liquid states.

Because NSL state represents a new class of quantum state which may be applied to learn the nature of the spin liquid state in other systems, for example, the Hubbard model on honeycomb lattice. People have proposed that quantum non-magnetic state of the Hubbard model on honeycomb lattice is really a Z_2 topological spin liquid ordered state, which is robust against arbitrary perturbation including the anisotropic term²⁸⁻³². However, our results of the quantum spin liquid state near the Mott transition of the π -flux Hubbard model are different - for the isotropic case, the non-magnetic state with $SU(2)$ spin rotation symmetry is nodal spin liquid (NSL) with gapless fermionic excitations and roton-like excitations; for the anisotropic case by adding arbitrary small easy-plane anisotropic term, the non-magnetic state becomes a Z_2 topological spin liquid with topological degenerate ground states. People may check the different theories by QMC approach in the future.

Finally, we give a comparison on different quantum orders with π -vortex (half-skyrmion or vison) and bosonic spinon. In general, for a system with π -vortex and bosonic spinon, there exist five types of quantum orders as⁶⁹:

1. *VBC state*: If one has gapless bosonic π -vortices and massive bosonic spinons, then the ground state

is always VBC state with spontaneous translation symmetry breaking;

2. *AF order*: If one has massive bosonic π -vortices and gapless bosonic spinons, then the ground state is an SDW order with spontaneous spin rotation symmetry breaking;
3. *Algebraic vortex liquid*: if one has gapless fermionic π -vortices and massive bosonic spinons, then ground state may be an algebraic vortex liquid (AVL)⁷⁰. In algebraic vortex liquid, fermionic excitations themselves couple to massless $U(1)$ gauge fields, as gives an example to algebraic spin liquid.
4. *Topological spin liquid*: if one has massive fermionic π -vortex and massive bosonic spinon, the ground state is a topological order with topologically degenerate ground state;
5. *Nodal spin liquid*: if one has gapless fermionic π -vortices (the fermionic spinons in this paper) and massive bosonic spinons, the ground state is a nodal spin liquid without any spontaneous symmetry breaking. In particular, there exist gapped roton-like gauge modes.

In order to giving a clear comparison, we give Table.2 :

	π -vortex	bosonic spinon
VBC state	massless Boson	massive Boson
AF order	massive Boson	massless Boson
AVL state	massless fermion	massive Boson
TSL order	massive Boson(fermion)	massive Boson
NSL state	massless fermion	massive Boson

We give a short remark on the relation between NSL state and the AVL in Ref.^{16,17,71}. In AVL, fermionic excitations coupling to a massless $U(1)$ gauge field and cannot be real quasi-particles. In NSL state, due to the protection from the mutual semion statistics between fermionic spinons and bosonic spinons, fermionic excitations are real excitations. There is neither bosonic spinon nor roton-like gauge mode in AVL state and the low energy effective theory of AVL state is also difference from that of NSL. In AVL, the spin-correlation shows critical behavior, while *in NSL state, although there exist gapless fermionic spinons, the spin-correlation decays exponentially*.

This research is supported by SRFDP, NFSC Grant no. 10874017 and National Basic Research Program of China (973 Program) under the grant No. 2011CB921803.

VI. APPENDIX A: EFFECTIVE $NL\sigma M$

In this appendix we use the path-integral formulation of electrons with spin rotation symmetry to obtain the

effective NL σ M of the spin fluctuations^{2,33,38–41,56}. The interaction term in Eq.[1] can be handled by using the SU(2) invariant Hubbard-Stratonovich decomposition in the arbitrary on-site unit vector $\mathbf{\Omega}_i$

$$\hat{n}_{i\uparrow}\hat{n}_{i\downarrow} = \frac{(\hat{c}_i^\dagger \hat{c}_i)^2}{4} - \frac{1}{4}[\mathbf{\Omega}_i \cdot \hat{c}_i^\dagger \sigma \hat{c}_i]^2. \quad (47)$$

Here $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. By replacing the electronic operators \hat{c}_i^\dagger and \hat{c}_j by Grassmann variables c_i^* and c_j , the effective Lagrangian of the 2D generalized Hubbard model at half filling is obtained:

$$\mathcal{L}_{\text{eff}} = \sum_i c_i^* \partial_\tau c_i - \sum_{\langle ij \rangle} (t_{i,j} c_i^* c_j + h.c.) - \Delta \sum_i c_i^* \mathbf{\Omega}_i \cdot \sigma c_i. \quad (48)$$

To describe the spin fluctuations, we use the Haldane's mapping^{36,37,39}:

$$\mathbf{\Omega}_i = (-1)^i \mathbf{n}_i \sqrt{1 - \mathbf{L}_i^2} + \mathbf{L}_i \quad (49)$$

where $\mathbf{n}_i = (n_i^x, n_i^y, n_i^z)$ is the Neel vector that corresponds to the long-wavelength part of $\mathbf{\Omega}_i$ with a restriction $\mathbf{n}_i^2 = 1$. \mathbf{L}_i is the transverse canting field that corresponds to the short-wavelength parts of $\mathbf{\Omega}_i$ with a restriction $\mathbf{L}_i \cdot \mathbf{n}_i = 0$. We then rotate $\mathbf{\Omega}_i$ to $\hat{\mathbf{z}}$ -axis for the spin indices of the electrons at i -site:^{2,33,38–41,56}

$$\begin{aligned} \psi_i &= U_i^\dagger c_i \\ U_i^\dagger \mathbf{n}_i \cdot \sigma U_i &= \sigma_z \\ U_i^\dagger \mathbf{L}_i \cdot \sigma U_i &= \mathbf{l}_i \cdot \sigma \end{aligned} \quad (50)$$

where $U_i \in \text{SU}(2)/\text{U}(1)$.

One then can derive the following effective Lagrangian after such spin transformation:

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \sum_i \psi_i^* \partial_\tau \psi_i + \sum_i \psi_i^* a_0(i) \psi_i \\ &- \sum_{\langle ij \rangle} (t_{i,j} \psi_i^* e^{ia_{ij}} \psi_j + h.c.) \\ &- \Delta \sum_i \psi_i^* \left[(-1)^i \sigma_z \sqrt{1 - \mathbf{l}_i^2} + \mathbf{l}_i \cdot \sigma \right] \psi_i \end{aligned} \quad (51)$$

where the auxiliary gauge fields $a_{ij} = a_{ij,1} \sigma_x + a_{ij,2} \sigma_y$ and $a_0(i) = a_{0,1}(i) \sigma_x + a_{0,2}(i) \sigma_y$ are defined as

$$e^{ia_{ij}} = U_i^\dagger U_j, \quad a_0(i) = U_i^\dagger \partial_\tau U_i. \quad (52)$$

In terms of the mean field result $M = (-1)^i \langle \psi_i^* \sigma_z \psi_i \rangle$ as well as the approximations,

$$\sqrt{1 - \mathbf{l}_i^2} \simeq 1 - \frac{\mathbf{l}_i^2}{2}, \quad e^{ia_{ij}} \simeq 1 + ia_{ij},$$

we obtain the effective Hamiltonian as:

$$\begin{aligned} \mathcal{L}_{\text{eff}} &\simeq \sum_i \psi_i^* \partial_\tau \psi_i + \sum_i \psi_i^* [a_0(i) - \Delta \sigma \cdot \mathbf{l}_i] \psi_i \\ &- \sum_{\langle ij \rangle} [t_{i,j} \psi_i^* (1 + ia_{ij}) \psi_j + h.c.] \\ &- \Delta \sum_i (-1)^i \psi_i^* \sigma_z \psi_i + \Delta M \sum_i \frac{\mathbf{l}_i^2}{2}. \end{aligned} \quad (53)$$

By integrating out the fermion fields ψ_i^* and ψ_i , the effective action with the quadric terms of $[a_0(i) - \Delta \sigma \cdot \mathbf{l}_i]$ and a_{ij} becomes

$$\mathcal{S}_{\text{eff}} = \frac{1}{2} \int_0^\beta d\tau \sum_i [-4\varsigma (a_0(i) - \Delta \sigma \cdot \mathbf{l}_i)^2 + 4\rho_s a_{ij}^2 + \frac{2\Delta^2}{U} \mathbf{l}_i^2]. \quad (54)$$

To give ρ_s and ς for calculation, we choose U_i to be

$$U_i = \begin{pmatrix} z_{i\uparrow}^* & z_{i\downarrow}^* \\ -z_{i\downarrow} & z_{i\uparrow} \end{pmatrix}, \quad (55)$$

where $\mathbf{n}_i = \bar{\mathbf{z}}_i \sigma \mathbf{z}_i$, $\mathbf{z}_i = (z_{i\uparrow}, z_{i\downarrow})^T$, $\bar{\mathbf{z}}_i \mathbf{z}_i = \mathbf{1}$. And the spin fluctuations around $\mathbf{n}_i = \hat{\mathbf{z}}_i$ are

$$\mathbf{n}_i = \hat{\mathbf{z}}_i + \text{Re}(\phi_i) \hat{\mathbf{x}} + \text{Im}(\phi_i) \hat{\mathbf{y}} \quad (56)$$

$$\mathbf{z}_i = \begin{pmatrix} 1 - |\phi_i|^2/8 \\ \phi_i/2 \end{pmatrix} + O(\phi_i^3). \quad (57)$$

Then the quantities $U_i^\dagger U_j$ and $U_i^\dagger \partial_\tau U_i$ can be expanded in the power of $\phi_i - \phi_j$ and $\partial_\tau \phi_i$,

$$U_i^\dagger U_j = e^{-i \frac{\phi_i - \phi_j}{2} \sigma_y} \quad (58)$$

$$U_i^\dagger \partial_\tau U_i = \begin{pmatrix} 0 & \frac{1}{2} \partial_\tau \phi_i \\ -\frac{1}{2} \partial_\tau \phi_i & 0 \end{pmatrix}. \quad (59)$$

According to Eq.(52), the gauge field a_{ij} and $a_0(i)$ are given as

$$a_{ij} = -\frac{1}{2} (\phi_i - \phi_j) \sigma_y \quad (60)$$

$$a_0(i) = \frac{i}{2} \partial_\tau \phi_i \sigma_y. \quad (61)$$

Supposing a_{ij} and $a_0(i)$ to be a constant in space and denoting $\partial_i \phi_i = \mathbf{a}$ and $\partial_\tau \phi_i = iB_y$, we have

$$a_{ij} = -\frac{1}{2} \mathbf{a} \cdot (\mathbf{i} - \mathbf{j}) \sigma_y \quad (62)$$

$$a_0(i) = -\frac{1}{2} B_y \sigma_y. \quad (63)$$

The energy of Hamiltonian of Eq.(54) becomes

$$E(B_y, \mathbf{a}) = \frac{1}{2} \zeta B_y^2 + \frac{1}{2} \rho_s \mathbf{a}^2. \quad (64)$$

Then one could get ζ and ρ_s from the following equations by calculating the partial derivative of the energy

$$\zeta = \frac{1}{N} \frac{\partial^2 E_0(B_y)}{\partial B_y^2} \Big|_{B_y=0} \quad (65)$$

$$\rho_s = \frac{1}{N} \frac{\partial^2 E_0(\mathbf{a})}{\partial \mathbf{a}^2} \Big|_{\mathbf{a}=0}. \quad (66)$$

Here $E_0(B_y)$ and $E_0(\mathbf{a})$ are the energy of the lower Hubbard band

$$E_0(B_y) = \sum_{\mathbf{k}} \left(E_{+,\mathbf{k}}^\zeta + E_{-,\mathbf{k}}^\zeta \right) \quad (67)$$

$$E_0(\mathbf{a}) = \sum_{\mathbf{k}} \left(E_{+,\mathbf{k}}^\rho + E_{-,\mathbf{k}}^\rho \right) \quad (68)$$

where $E_{+,\mathbf{k}}^\zeta$, $E_{-,\mathbf{k}}^\zeta$ and $E_{+,\mathbf{k}}^\rho$, $E_{-,\mathbf{k}}^\rho$ are the energies of the following Hamiltonian \mathcal{H}^ζ and \mathcal{H}^ρ

$$\begin{aligned} \mathcal{H}^\zeta = & - \sum_{\langle ij \rangle} (t_{i,j} \psi_i^* \psi_j + h.c.) - \Delta \sum_i (-1)^i \psi_i^* \sigma_z \psi_i \\ & + \sum_i \psi_i^* a_0(i) \psi_i \end{aligned} \quad (69)$$

$$\mathcal{H}^\rho = - \sum_{\langle ij \rangle} (t_{i,j} \psi_i^* e^{a_{ij}} \psi_j + h.c.) - \Delta \sum_i (-1)^i \psi_i^* \sigma_z \psi_i. \quad (70)$$

Using the Fourier transformations for \mathcal{H}^ζ , we have the spectrum of the lower band of \mathcal{H}^ζ

$$E_{\pm,\mathbf{k}}^\zeta = -\sqrt{\left(|\xi_{\mathbf{k}}| \pm \frac{B_y}{2} \right)^2 + \Delta^2} \quad (71)$$

where $\xi_{\mathbf{k}} = \pm \sqrt{4t^2 (\cos^2 k_x + \cos^2 k_y)}$. And ζ is obtained as

$$\zeta = \frac{1}{N} \sum_{\mathbf{k}} \frac{\Delta^2}{4(|\xi_{\mathbf{k}}|^2 + \Delta^2)^{\frac{3}{2}}}. \quad (72)$$

Similarly, using the Fourier transformations for \mathcal{H}^ρ , we obtain the spectrum of the lower band of \mathcal{H}^ρ

$$E_{\pm,\mathbf{k}}^\rho = -\sqrt{\Delta^2 + |\vartheta|^2 + |\varphi|^2 \pm \left[4\Delta^2 |\vartheta|^2 - (\varphi\vartheta^* - \vartheta\varphi^*)^2 \right]^{\frac{1}{2}}} \quad (73)$$

where φ and ϑ are defined as

$$\varphi = -t \sum_{\delta} e^{i\mathbf{k} \cdot \delta} \cos \left(\frac{1}{2} \mathbf{a} \cdot \delta \right) \quad (74)$$

$$\vartheta = -t \sum_{\delta} e^{i\mathbf{k} \cdot \delta} \sin \left(\frac{1}{2} \mathbf{a} \cdot \delta \right) \quad (75)$$

where $\delta = a(e_x, e_y)$ and $e_x^2 = e_y^2 = 1$. Using Eq.(66), ρ_s is given as

$$\rho_s = \frac{1}{N} \sum_{\mathbf{k}} \frac{\epsilon^2}{4(|\xi_{\mathbf{k}}|^2 + \Delta^2)^{\frac{3}{2}}}. \quad (76)$$

For the π -flux Hubbard model, we get the corresponding coefficient ϵ^2 as

$$\begin{aligned} \epsilon^2 = & t^2 [\cos(2k_x) (\Delta^2 + 8t^2 + 4t^2 \cos(2k_y)) \\ & + \Delta^2 + 3t^2 + t^2 \cos(4k_x)]. \end{aligned} \quad (77)$$

Next, to learn the properties of the low energy physics, we study the continuum theory of the effective action in Eq.(54). In the continuum limit, we denote \mathbf{n}_i , \mathbf{l}_i , $ia_{ij} \simeq U_i^\dagger U_j - 1$ and $a_0(i) = U_i^\dagger \partial_\tau U_i$ by $\mathbf{n}(x, y)$, $\mathbf{l}(x, y)$, $U^\dagger \partial_x U$ (or $U^\dagger \partial_y U$) and $U^\dagger \partial_\tau U$, respectively. From the relations between $U^\dagger \partial_\mu U$ and $\partial_\mu \mathbf{n}$,

$$a_\tau^2 = a_{\tau,1}^2 + a_{\tau,2}^2 = -\frac{1}{4} (\partial_\tau \mathbf{n})^2, \quad \tau = 0, \quad (78)$$

$$a_\mu^2 = a_{\mu,1}^2 + a_{\mu,2}^2 = \frac{1}{4} (\partial_\mu \mathbf{n})^2, \quad \mu = x, y, \quad (79)$$

$$\mathbf{a}_0 \cdot \mathbf{l} = -\frac{i}{2} (\mathbf{n} \times \partial_\tau \mathbf{n}) \cdot \mathbf{l}, \quad (80)$$

the continuum formulation of the action in Eq.(54) turns into

$$\begin{aligned} \mathcal{S}_{\text{eff}} = & \frac{1}{2} \int_0^\beta d\tau \int d^2 \mathbf{r} [\zeta (\partial_\tau \mathbf{n})^2 + \rho_s (\nabla \mathbf{n})^2 \\ & - 4i\Delta \zeta (\mathbf{n} \times \partial_\tau \mathbf{n}) \cdot \mathbf{l} + \left(\frac{2\Delta^2}{U} - 4\Delta^2 \zeta \right) \mathbf{l}^2] \end{aligned} \quad (81)$$

where the vector \mathbf{a}_0 is defined as $\mathbf{a}_0 = (a_{0,1}, a_{0,2}, 0)$.

Finally we integrate the transverse canting field \mathbf{l} and obtain the effective NL σ M as

$$\mathcal{S}_{\text{eff}} = \frac{1}{2g} \int_0^\beta d\tau \int d^2 \mathbf{r} \left[\frac{1}{c} (\partial_\tau \mathbf{n})^2 + c (\nabla \mathbf{n})^2 \right] \quad (82)$$

with a constraint $\mathbf{n}^2 = 1$. The coupling constant g and spin wave velocity c are defined as^{2,33,38-41,56}:

$$g = \sqrt{\frac{1}{\chi^\perp \rho_s}}, \quad c^2 = \frac{\rho_s}{\chi^\perp} \quad (83)$$

where χ^\perp is the transverse spin susceptibility

$$\chi^\perp = \left[\left(\frac{1}{N} \sum_{\mathbf{k}} \frac{\Delta^2}{4(|\xi_{\mathbf{k}}|^2 + \Delta^2)^{\frac{3}{2}}} \right)^{-1} - 2U \right]^{-1}. \quad (84)$$

Then we use the effective NL σ M to study the magnetic properties of the insulator state. The Lagrangian of NL σ M with a constraint ($\mathbf{n}^2 = 1$) by a Lagrange multiplier λ becomes

$$\mathcal{L}_{\text{eff}} = \frac{1}{2cg} \left[(\partial_\tau \mathbf{n})^2 + c^2 (\nabla \mathbf{n})^2 + i\lambda (1 - \mathbf{n}^2) \right] \quad (85)$$

where $i\lambda = m_s^2$ and m_s is the mass gap of the spin fluctuations. Using the large- N approximation we rescale the field $\mathbf{n} \rightarrow \sqrt{N} \mathbf{n}$ and obtain the saddle-point equation of motion as⁴⁴⁻⁴⁶

$$(n_0)^2 + k_B T \sum_{\omega_n, \mathbf{q} \neq \mathbf{0}} \frac{gc}{\omega_n^2 + c^2 \mathbf{q}^2 + m_s^2} = 1. \quad (86)$$

In Eq.(86), n_0 is the mean field value of \mathbf{n} and $\omega_n = 2\pi n k_B T$, $n = \text{integers}$.

From Eq.(86), we may get the solution of m_s as

$$m_s = 2k_B T \sinh^{-1} \left[e^{-\frac{2\pi c}{g k_B T}} \sinh \left(\frac{c\Lambda}{2k_B T} \right) \right]. \quad (87)$$

At zero temperature the solutions of n_0 and m_s of Eq.(86) are determined by the coupling constant g . There exists a critical point $g_c = \frac{4\pi}{\Lambda}$: For the case of $g < \frac{4\pi}{\Lambda}$, we get a non-magnetic insulator with solutions of n_0 and m_s :

$$n_0 = (1 - \frac{g}{g_c})^{1/2}, m_s = 0 \quad (88)$$

For the case of $g > \frac{4\pi}{\Lambda}$, we get a long range AF order with solutions of n_0 and m_s :

$$n_0 = 0, m_s = 4\pi c(\frac{1}{g_c} - \frac{1}{g}) \quad (89)$$

VII. APPENDIX B : THE INDUCED KINETIC TERM OF GAUGE FIELD

Starting from the CP(1) model,

$$\mathcal{L}_s = \frac{1}{2g} [|(\partial_\mu - ia_\mu)\mathbf{z}|^2 + m_z^2 \mathbf{z}^2],$$

we calculate the induced the kinetic term of the gauge field a_μ .

We obtain the expression of the expansion of the renormalization 2-point function at one-loop order :

$$\frac{2}{(2\pi)^3} \int d^3k \left[\frac{(k_\mu + 2p_\mu)(k_\nu + 2p_\nu)}{(k^2 + m_z^2)((k+p)^2 + m_z^2)} - 2\delta_{\mu\nu} \frac{1}{k^2 + m_z^2} \right] \quad (90)$$

At low energy limit as $p^2 \rightarrow 0$, we simplify the above integral to the following one

$$\begin{aligned} & \frac{2}{(2\pi)^3} \pi^{1/2} \Gamma\left(\frac{1}{2}\right) \frac{1}{3m_z^2} (p_\mu p_\nu - \delta_{\mu\nu} p^2) \quad (91) \\ & = \frac{2}{24\pi^2 m_z^2} (p_\mu p_\nu - \delta_{\mu\nu} p^2). \end{aligned}$$

At low energy limit ($p^2 \rightarrow 0$). Thus from $-\frac{1}{4e_a^2}(\partial_\nu a_\mu)^2$, the induced coupling constants of three dimensional gauge field is obtained as

$$e_a^2 = 3\pi m_z^2. \quad (92)$$

Using similar approach, oen may get the induced kinetic term of gauge field A_μ .

VIII. APPENDIX C : PROJECTIVE SYMMETRY GROUP OF HALF SKYRMIONS

In the insulator state (non-magnetic or magnetic order), there exists a bosonic spinon on each site. Because, the half skyrmion is really the vortex, we may consider the dynamics of half skyrmions as that of the vortices of

the insulator state of bosons on lattice with unit filling (See Fig.7). As a result, the square lattice of bosonic spinons plays a role of π -flux phase of half skyrmion on dual lattice $\mathbf{I} = (I_x, I_y)$. So we may use the approach of

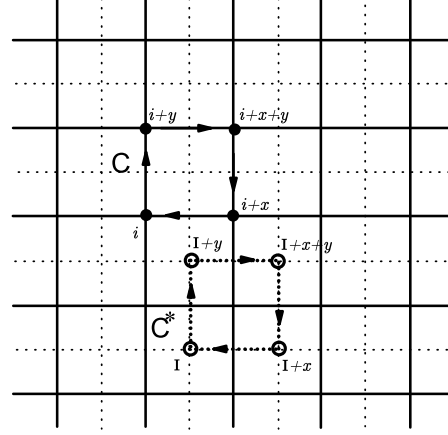


FIG. 7: Spin-vortex on dual lattice

the projective representation of the space group (PSG) for vortex to learn the dynamics of half skyrmion here. The PSG denotes the phase transformations associating the operations in the space group.

For the π -flux phase of half skyrmions, we have x and y translations, and a $\pi/2$ rotation of the PSG on the dual square lattice as

$$\begin{aligned} T_y &: f(I_x, I_y) \rightarrow f(I_x, I_y - 1) \\ T_x &: f(I_x, I_y) \rightarrow f(I_x + 1, I_y)(-1)^{I_y} \\ R_{\pi/2} &: f(I_x, I_y) \rightarrow f(I_x, -I_y)(-1)^{I_x + I_y} \quad (93) \end{aligned}$$

where $f(I_x, I_y)$ is the wave-function of half skyrmions and $\mathbf{I} = (I_x, I_y)$ are the dual lattices. One may check that the operations associated with translations in the PSG do not commute as $T_x T_y = -T_y T_x$ that means the half skyrmion obtains a π phase encircling a site of the direct lattice (i) with one bosonic spinon.

We are looking at the half skyrmions hopping around dual square lattice in the presence of π -flux per plaquette. Then the effective action should be invariant under PSG and the dual $U(1)$ gauge symmetry. For the simplest gauge and nearest neighbor hopping, the effective Hamiltonian of half skyrmions is

$$\mathcal{H}_{hs} = - \sum_{\langle \mathbf{I}, \mathbf{J} \rangle} (\tilde{t}_{\mathbf{I}, \mathbf{J}} f_{\mathbf{I}}^\dagger f_{\mathbf{J}} + h.c.) \quad (94)$$

where the effective nearest-neighbor hopping $\tilde{t}_{\mathbf{I}, \mathbf{J}}$ could be chosen as $\tilde{t}_{\mathbf{i}, \mathbf{i}+\hat{x}} = t$, $\tilde{t}_{\mathbf{i}, \mathbf{i}+\hat{y}} = \tilde{t}e^{\pm i\frac{\pi}{2}}$.

-
- * Corresponding author; Electronic address: spkou@bnu.edu.cn
- ¹ P. Fazekas and P.W. Anderson, *Philos. Mag.* **30**, 432 (1974); P. W. Anderson, *Science* **235**, 1196 (1987).
 - ² X. G. Wen, *Quantum Field Theory of Many-Body Systems*, (Oxford Univ. Press, Oxford, 2004).
 - ³ Chen Zeng and Veit Elser, *Phys. Rev. B* **42**, 8436 (1990).
 - ⁴ S. Sachdev, *Phys. Rev. B* **45**, 12377(1992).
 - ⁵ S. Ryu, O. I. Motrunich, J. Alicea, Matthew P. A. Fisher, *Phys. Rev. B* **75**, 184406 (2007).
 - ⁶ Michael Hermele, Ying Ran, Patrick A. Lee, and Xiao-Gang Wen, *Phys. Rev. B* **77**, 224413 (2008).
 - ⁷ H. C. Jiang, Z. Y. Weng, D. N. Sheng, *Phys. Rev. Lett.* **101**, 117203 (2008).
 - ⁸ V.N. Kotov *et al.*, *Phil. Mag. B* **80**, 1483 (2000).
 - ⁹ E. Dagotto and A. Moreo, *Phys. Rev. B* **39**, 4744 (1989); *Phys. Rev. Lett.* **63**, 2148, (1989).
 - ¹⁰ K. Sano *et al.*, *J. Phys. Soc. Jpn.* **60**, 3807 (1991).
 - ¹¹ H. J. Schultz and T. A. Ziman, *EPL* **18**, 355 (1992).
 - ¹² S. Sorella, *Phys. Rev. Lett.* **80**, 4558 (1998).
 - ¹³ Luca Capriotti, *et al.*, *Phys. Rev. Lett.* **87**, 097201 (2001).
 - ¹⁴ K. Takano, Y. Kito, Y. Ono, K. Sano, *Phys. Rev. Lett.* **91**, 197202 (2003).
 - ¹⁵ G. M. Zhang, H. Hu, L. Yu, *Phys. Rev. Lett.*, **91**, 067201 (2003).
 - ¹⁶ X. G. Wen, *Phys. Rev. B* **65**, 165113 (2002).
 - ¹⁷ M. Hermele, T. Senthil, and M. P. A. Fisher, *Phys. Rev. B* **72**, 104404 (2005).
 - ¹⁸ X. G. Wen, *Phys. Rev. B* **44**, 2664 (1991).
 - ¹⁹ X. G. Wen, F. Wilczek and A. Zee, *Phys. Rev. B* **39**, 11413 (1989).
 - ²⁰ X. G. Wen, *Phys. Rev. B* **40**, 7387 (1989).
 - ²¹ Y. Shimizu, *et al.*, *Phys. Rev. Lett.* **91**, 107001 (2003).
 - ²² A. Kawamoto, Y. Honma, and K. I. Kumagai, *Phys. Rev. B* **70**, 060510(R) (2004).
 - ²³ Y. Kurosaki, *et al.*, *Phys. Rev. Lett.* **95**, 177001 (2005).
 - ²⁴ S. S. Lee and P. A. Lee, *Phys. Rev. Lett.* **95**, 036403 (2005).
 - ²⁵ Michael Hermele, *Phys. Rev. B* **76**, 035125 (2007).
 - ²⁶ Meng Z Y, Lang T C, Wessel S, Assaad F F, Muramatsu A, arXiv:1003.5809.
 - ²⁷ G. Y. Sun and S. P. Kou, arXiv:0911.3002.
 - ²⁸ F. Wang, *Phys. Rev. B* **82**, 024419 (2010).
 - ²⁹ Y.-M. Lu and Y. Ran, arXiv:1005.4229; Y.-M. Lu and Y. Ran, arXiv:1007.3266.
 - ³⁰ B. K. Clark, D. A. Abanin, and S. L. Sondhi, arXiv:1010.3011.
 - ³¹ G. Wang, M. O. Goerbig, B. Gremaud, and C. Miniatura, arXiv:1006.4456.
 - ³² A. Vaezi and X.-G. Wen, arXiv:1010.5744.
 - ³³ G. Y. Sun and S. P. Kou, *EPL*, **87**, 67002 (2009).
 - ³⁴ T. C. Hsu, *Phys. Rev. B* **41**, 11379 (1990).
 - ³⁵ The π -flux Hubbard model (or the Hubbard model with ϕ flux) is not a physics model in condensed matter physics. However, people pointed out that the π -flux Hubbard model may be designed with ultracold atoms in an optical lattice. An artificial magnetic field of π -flux (or ϕ flux) in an optical square lattice is proposed to be realized by different approaches recently⁷³⁻⁷⁵.
 - ³⁶ F.D.M. Haldane, *Phys. Lett.* **93A**, 464(1983).
 - ³⁷ A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer-Verlag, New York, 1994).
 - ³⁸ N. Dupuis, *Phys. Rev. B* **65**, 245118 (2002).
 - ³⁹ K. Borejsza, N. Dupuis, *Euro Phys. Lett.* **63**, 722 (2003); K. Borejsza and N. Dupuis *Phys. Rev. B* **69**, 085119 (2004).
 - ⁴⁰ H. J. Schulz, *Phys. Rev. Lett.* **65**, 2462(1990); H. J. Schulz, in *The hubbard Model*, edited by D. Baeriswyl(Plenum, New York, 1995).
 - ⁴¹ Z. Y. Weng, C. S. Ting, and T. K. Lee, *Phys. Rev. B* **43**, 3790 (1991).
 - ⁴² The effective O(3) nonlinear σ model is derived by integrating the fermionic field. We find that there is induced Wess-Zumino term here. So the effective O(3) nonlinear σ model don't have a Berry term as that appears in the effective model from the Heisenberg model.
 - ⁴³ D.P. Arovas and A. Auerbach, *Phys. Rev. B* **38**, 316 (1988); A. Auerbach, *Interacting Electrons and Quantum Magnetism*, (Springer-Verlag New York, Inc. 1994).
 - ⁴⁴ S. Chakravarty, *et al.*, *Phys. Rev. B* **39**, 2344 (1989).
 - ⁴⁵ A. V. Chubukov and S. Sachdev, J. Ye, *Phys. Rev. B* **49**, 11919 (1994).
 - ⁴⁶ S. Sachdev, *Quantum Phase Transitions*, Cambridge University Press (1999).
 - ⁴⁷ M. Oshikawa, *Phys. Rev. Lett.* **84**, 1535 (2000).
 - ⁴⁸ M. B. Hastings, *Phys. Rev. B* **69** 10443 (2004).
 - ⁴⁹ A. A. Belavin and A. M. Polyakov, *JETP Lett.* **22**, 245 (1975).
 - ⁵⁰ J. A. Vergés, E. Louis, P. S. Lomdahl, F. Guinea, and A. R. Bishop, *Phys. Rev. B* **43**, 6099, (1991); S. John, M. Berciu and A. Golubentsev, *Europhys. Lett.* **41**, 31 (1998); M. Berciu and S. John, *Phys. Rev. B* **57**, 9521 (1998); M. Berciu and S. John, *Phys. Rev. B* **61**, 16454 (2000).
 - ⁵¹ T. Morinari, *Phys. Rev. B* **72**, 104502 (2005).
 - ⁵² T. K. Ng, *Phys. Rev. B* **52**, 9491 (1995); *Phys. Rev. Lett.* **82**, 3504 (1999); *Int. J. Mod. Phys. B* **14**, 349 (2000).
 - ⁵³ Y. Otsuka and Y. Hatsugai, *Phys. Rev. B* **65**, 073101 (2002).
 - ⁵⁴ Z. Y. Weng, D. N. Sheng, and C. S. Ting, *Phys. Rev. Lett.* **80**, 5401 (1998); Zheng-Yu Weng, *Int. J. Mod. Phys. B* **21**, 773 (2007).
 - ⁵⁵ S. P. Kou and Z. Y. Weng, *Phys. Rev. Lett.* **90**, 157003 (2003).
 - ⁵⁶ S. P. Kou, *Phys. Rev. B* **78**, 233104 (2008).
 - ⁵⁷ A. Auerbach, *et al.*, *Phys. Rev. B* **43**, 11515 (1991).
 - ⁵⁸ G. Kotliar, *Phys. Rev. B* **37**, 3664 (1988).
 - ⁵⁹ T. C. Hsu, *Phys. Rev. B* **41**, 11379 (1990).
 - ⁶⁰ I. Affleck and J. B. Marston, *Phys. Rev. B* **37**, 3774 (1988); J. B. Marston and I. Affleck, *Phys. Rev. B* **39**, 11538 (1989).
 - ⁶¹ M. Carena, S. Chaudhuri, and C. E. Wagner, *Phys. Rev. D* **42**, 2120 (1990).
 - ⁶² R. Jackiw and C. Rebbi, *Phys. Rev. D* **13**, 3398 (1976).
 - ⁶³ S. Sachdev, K. Park, *Ann. of Phys.*, **298**, 58 (2002).
 - ⁶⁴ T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, *Phys. Rev. B* **70**, 144407 (2004).
 - ⁶⁵ S. P. Kou, X. L. Qi, Z. Y. Weng, *Phys. Rev. B* **71**, 235102 (2005); X. L. Qi, Z. Y. Weng, *Phys. Rev. B* **76**, 104502 (2007).
 - ⁶⁶ S. P. Kou, M. Levin, X. G. Wen, *Phys. Rev. B* **78**, 155134 (2008).
 - ⁶⁷ A.M. Polyakov, *Nucl. Phys. B* **120**, 429 (1977). A.M. Polyakov, *Gauge fields and strings* (Harwood Academic

- Publishers, London, 1987).
- ⁶⁸ S. P. Kou, T. Li, Z. Y. Weng, EPL. **88**, 17010 (2009).
 - ⁶⁹ C. K. Xu and S. Sachdev, cond/mat-0811.1220.
 - ⁷⁰ J. Alicea, O. I. Motrunich, M. Hermele, M. P. A. Fisher, Phys. Rev. **B 72** (2005) 064407; J. Alicea, O. I. Motrunich, M. P. A. Fisher, Phys. Rev. Lett. 95, 247203 (2005); S. Ryu, O. I. Motrunich, J. Alicea, M. P. A. Fisher, Phys. Rev. **B 75**, 184406 (2007).
 - ⁷¹ P. Ghaemi and T. Senthil, Phys. Rev. B **73**, 054415 (2006).
 - ⁷² L. Balents, M. P. A. Fisher and C. Nayak, Int. J. Mod. Phys. B **12**, 1033 (1998).
 - ⁷³ D. Jaksch and P. Zoller, New J. Phys. 5, 56 (2003).
 - ⁷⁴ G. Juzeliūnas and P. Öhberg, Phys. Rev. Lett. 93, 033602 (2004); G. Juzeliūnas, P. Öhberg, J. Ruseckas and A. Klein, Phys. Rev. A 71, 053614 (2005).
 - ⁷⁵ Y. J. Lin, *et. al.*, Phys. Rev. Lett. 102, 130401 (2009).